Section 10.3

Series

Definition 10.3.1. Series
Given a sequence $\{a_1, a_2, a_3, \dots\}$, the sum of its terms

$$a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k$$

is called an *infinite series*. The *sequence of partial sums* $\{S_n\}$ associated with this series has the terms

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ S_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k \end{aligned}$$

If the sequence of partial sums $\{S_n\}$ has a limit L, then we say the infinite series converges to L. We write

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n = L.$$

If the sequence of partial sums diverges, then the infinite series also diverges.

Geometric Series

Geometric series are the sums of geometric sequences, their partial sums are called *geometric sums*. For the sequence $\{ar^n\}_{n=0}^{\infty}$, the geometric sums have the form

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \sum_{k=0}^{n-1} ar^k,$$

where $a \neq 0$ and r is a real number.

Then $\sum_{k=0}^{\infty} ar^k = \lim_{n\to\infty} S_n = \lim_{n\to\infty} a\frac{1-r^n}{1-r}$. For what values of r does this limit converge?

Example 10.3.2. Evaluate the following series or state that they diverge.

1. $\sum_{k=0}^{\infty} 1.1^k$

2. $\sum_{k=2}^{\infty} 3(-0.75)^k$

3. $1 + 0.035 + 0.00035 + 0.0000035 + \cdots$

Telescoping Series

Example 10.3.3. 1. $\displaystyle\sum_{k=1}^{\infty}\left(\cos\frac{\pi}{k}-\cos\frac{\pi}{k+1}\right)$

2. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2-k}$

Theorem 10.3.4. Properties of Convergent Series

1. Suppose $\sum a_k$ converges to A and c is a real number. The series $\sum ca_k$ converges and $\sum ca_k = c\sum a_k = cA$.
2. Suppose $\sum a_k$ diverges. Then $\sum ca_k$ also diverges for any real number $c \neq 0$.
3. Suppose $\sum a_k$ converges to A and $\sum b_k$ converges to B. The series $\sum \left(a_k \pm b_k\right)$ converges, and $\sum \left(a_k \pm b_k\right) = \sum a_k \pm \sum b_k = A \pm B$.
4. Suppose $\sum a_k$ diverges and $\sum b_k$ converges. Then $\sum \left(a_k \pm b_k\right)$ diverges.
5. If M is a positive integer, then $\displaystyle\sum_{k=1}^{\infty} a_k$ and $\displaystyle\sum_{k=M}^{\infty} a_k$ either both converge or both diverge.

Example 10.3.5. Evaluate each series or state that it diverges.

1. $\displaystyle\sum_{k=1}^{\infty}\left(-\frac{2}{3}\right)^{k}$

2. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{(k+6)(k+7)}$

3. $\displaystyle\sum_{k=0}^{\infty}\frac{2-3^{k}}{6^{k}}$

4. $\displaystyle\sum_{k=2}^{\infty}\frac{\ln((k+1)k^{-1})}{(\ln k)\ln(k+1)}$

5. $\displaystyle\sum_{k=1}^{\infty}\frac{3^{k}}{(3^{k+1}-1)(3^{k}-1)}$

6. $\displaystyle\sum_{k=1}^{\infty}\left(2\left(\frac{3}{5}\right)^{k}+3\left(\frac{4}{9}\right)^{k}\right)$